NONLINEAR MODEL PREDICTIVE CONTROL WITH CONSTRAINT AGGREGATION

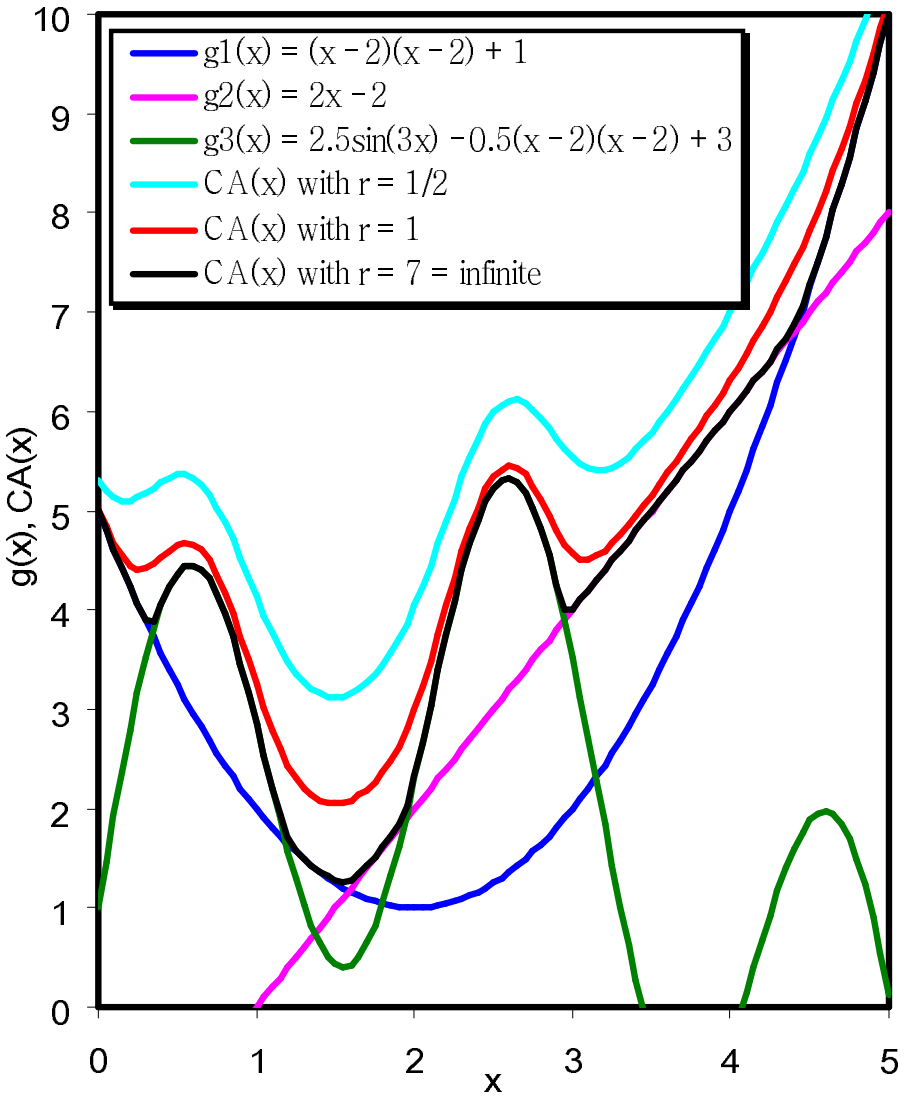
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CONSTRAINT AGGREGATION



Constraints : $g_j(\mathbf{x}) \leq 0, \quad j = 1, \ldots, J$

Aggregation Function :

$$CA(g_j(\mathbf{x})) = \frac{1}{\rho} \ln\left[\sum_j^J \exp\left(\rho\, g_j(x)\right)\right] \leq 0$$

$$\rho > 0$$

- Also known as the **KS** (*Kreisselmeier & Steinhauser*) function, which is an **overestimator** for the vector of inequality constraints of the form $\mathbf{g(x)} \leq \mathbf{0}$.
- All *J* constraints of the form $\mathbf{g(x)} \leq \mathbf{0}$ are lumped into a single constraint, $CA(\mathbf{x}) \leq 0$.

Some Properties: (Biegler et al., *Comput.Chem.Eng.*, 2000)

$$CA(\mathbf{x},\rho) \geq max_j \{g_j(\mathbf{x})\}$$

$$\lim_{\rho \to \infty} CA(\mathbf{x},\rho) = max_j \{g_j(\mathbf{x})\}$$

$$CA(\mathbf{x},\rho_2) \geq CA(\mathbf{x},\rho_1) \text{ such that } \rho_1 > \rho_2$$

OPTIMIZATION WITH CONSTRAINT AGGREGATION

- **without constraint aggregation**

$$\begin{aligned} \underset{\mathbf{x}}{\text{minimize}} \quad & f(\mathbf{x}) \\ \text{s.t.} \quad & g_j(\mathbf{x}) \le 0 \quad j = 1, \ldots, J \\ & h_k(\mathbf{x}) = 0 \quad k = 1, \ldots, K \\ & \mathbf{x}^- \le \mathbf{x} \le \mathbf{x}^+ \end{aligned}$$



- **with constraint aggregation**

$$\begin{aligned} \underset{\mathbf{x}}{\text{minimize}} \quad & f(\mathbf{x}) \\ \text{s.t.} \quad & CA(\mathbf{x}) \le 0 \\ & \mathbf{x}^- \le \mathbf{x} \le \mathbf{x}^+ \end{aligned}$$

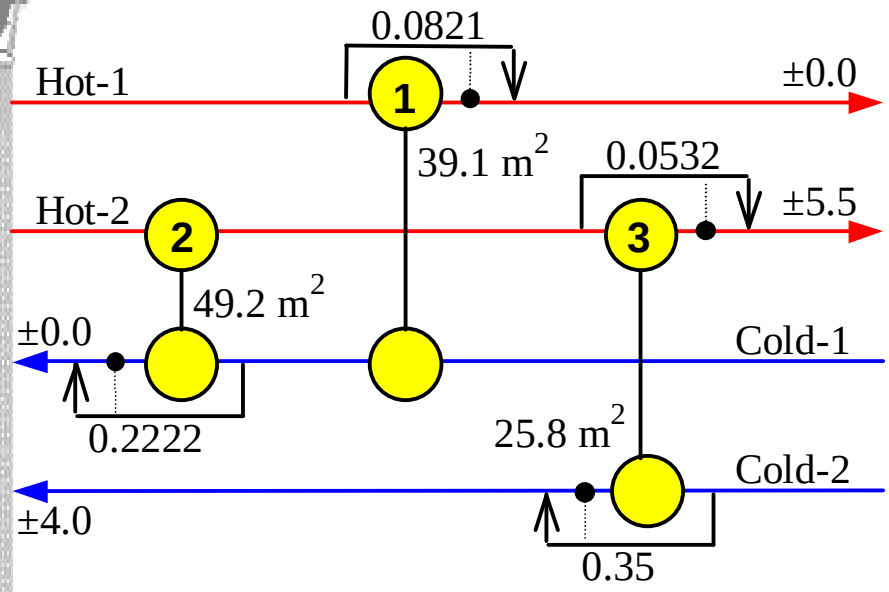
where : $CA(g_m(\mathbf{x})) = \frac{1}{\rho}\ln\left[\sum_{m}^{M}\exp\left(\rho\, g_m(x)\right)\right] \quad m = 1, \ldots, J+2K$

$$\begin{aligned} g_m(\mathbf{x}) : \quad & g_j(\mathbf{x}) \le 0 , \quad m = 1, \quad \ldots, J \\ g_m(\mathbf{x}) : \quad & h_k(\mathbf{x}) \le 0 , \quad m = J+1, \quad \ldots, J+K \\ g_m(\mathbf{x}) : \quad & -h_k(\mathbf{x}) \le 0 , \quad m = J+K+1, \quad \ldots, J+2K \end{aligned}$$

However : Even if all **g** & **h** are linear,
CA is nonlinear !



RETROFIT HEAT-EXCHANGER NETWORK



U_n = 0.5 kW/m^2-K

Stream	W (kW/K)	T_{in} (K)	T_{out} (K)
Hot-1	10	620	385
Hot-2	15	720	400
Cold-1	20	300	560
Cold-2	30	280	340

Distributed-Parameter (PDE) Model :

$$\frac{\partial T_H}{\partial t} = -v_H \frac{\partial T_H}{\partial z} + \frac{h_H a_H h_C a_C}{h_C a_C + h_H a_H} \frac{(T_C - T_H)}{A_H \rho_H C_{p_H}}$$

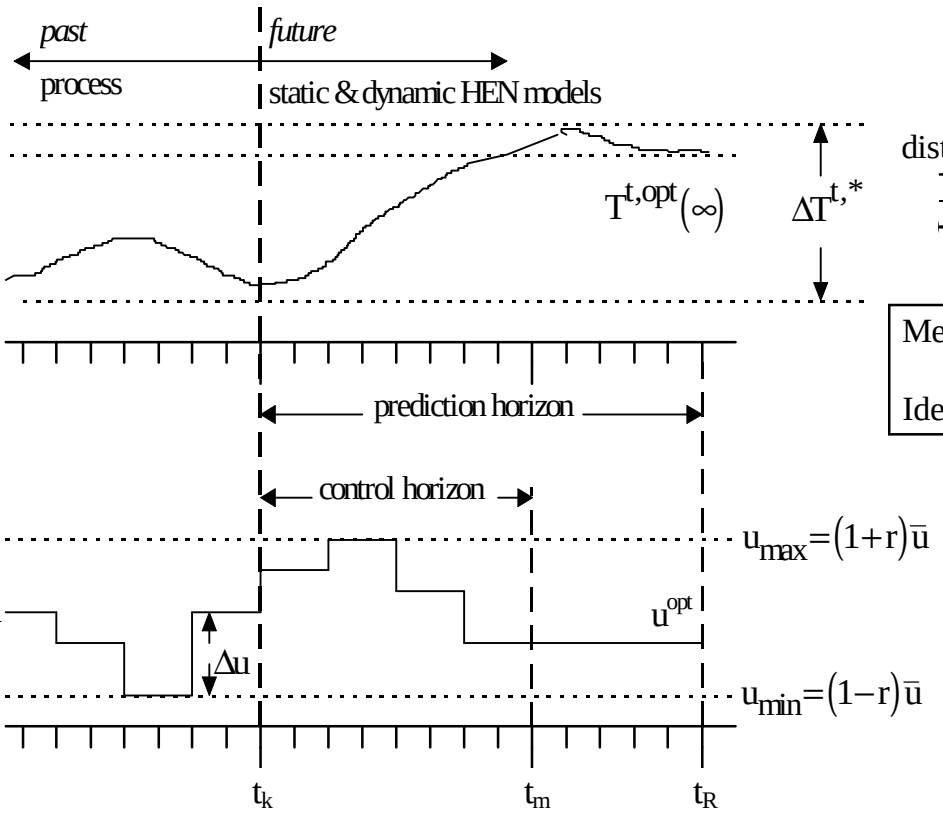
$$\frac{\partial T_C}{\partial t} = +v_C \frac{\partial T_C}{\partial z} + \frac{N_T h_C a_C h_H a_H}{h_C a_C + h_H a_H} \frac{(T_H - T_C)}{A_C \rho_C C_{p_C}}$$

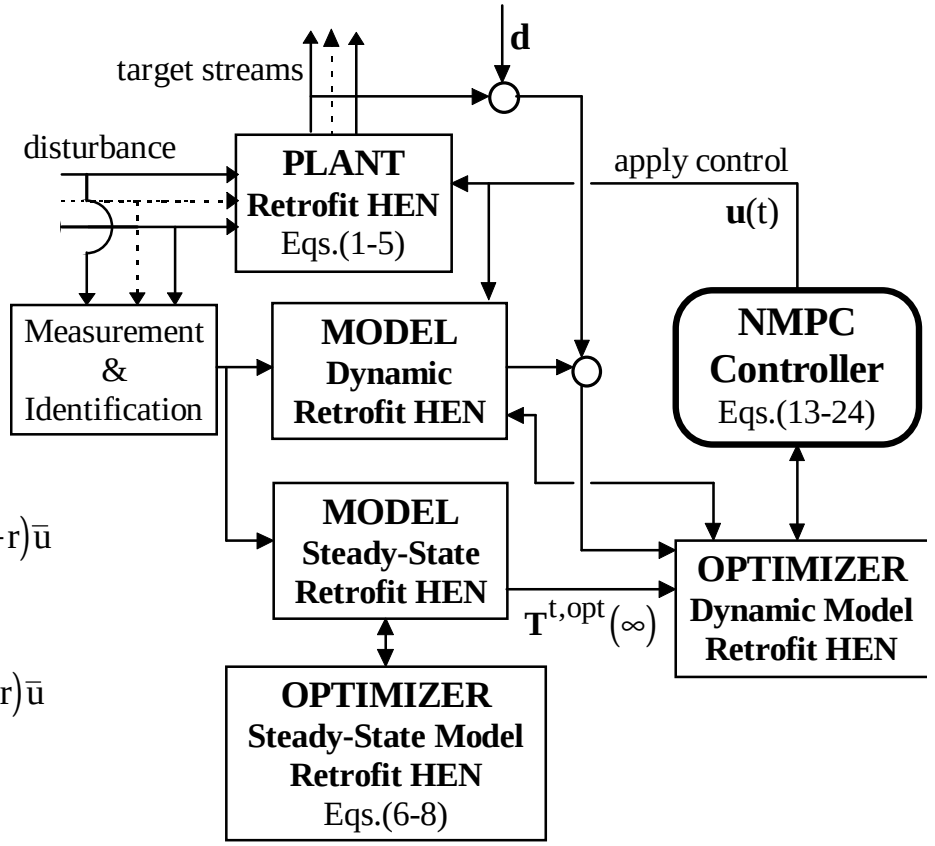
bypass mixing cold side: $T_C^t = T_C^o u_C + T_C(1 - u_C)$

bypass mixing hot side: $T_H^t = T_H^o u_H + T_H(1 - u_H)$

- Single-tube-pass, **shell-and-tube** exchangers.
- Distributed-parameter model of the heat exchangers were used to construct the dynamic model of the HENs by connecting the PDEs of the individual exchangers through boundary conditions.
- Utilizing the **method of lines** numerical technique, these PDEs were discretized in z-direction and converted into ODEs using **five-point biased-upwind** finite-difference formulas, which ***eliminated steady-state offset*** problem.

GENERIC NMPC OF RETROFIT HEN





Some NMPC Parameters

$\Delta t = t_{k+1} - t_k = 30$ s	m = 3 steps	R = 4 steps
$t_R - t_m = \Delta t$	r = 0.25	r* = 0.1

FORMULATION OF GENERIC NMPC OF RETROFIT HEN

$$\underset{\mathbf{u}(i),\ldots,\mathbf{u}(i+m-1)}{\textit{minimize}} \quad \Phi(\mathbf{u}(i)) = \int_{t_k}^{t_{k+R}} \left\| \mathbf{T}_k^{t,opt}(\infty) - \mathbf{T}_{MP}^{t}(i) \right\| dt \qquad i=k,\ldots,k+R$$

$$\mathbf{T}_{MP}^{t}(i) = \mathbf{T}_{M}^{t}(i) + \mathbf{d}(k) \qquad i=k,\ldots,k+R$$

$$\mathbf{d}(k) = \mathbf{T}_{P}^{t}(k) - \mathbf{T}_{M}^{t}(k)$$

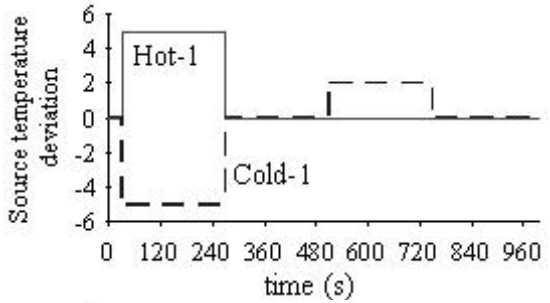
$$\dot{\mathbf{x}}(t,z) = \boldsymbol{f}\left(\mathbf{x}(t,z), \partial\mathbf{x}(t,z)/\partial z, \mathbf{u}(t), \mathbf{p}, \mathbf{d}\right) \qquad t \in [k,\ldots,k+R]$$

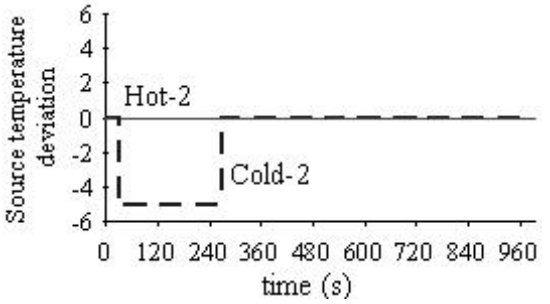
$$\mathbf{x}(k,z) = \bar{\mathbf{x}}(z)$$

$$\Delta\mathbf{u}(i+1) = \mathbf{u}(i+1) - \mathbf{u}(i)$$

$$\mathbf{u}(i) = \mathbf{u}(k+m-1) \qquad \forall\ i > k+m-1$$

$$x(t)\big|_{k+R} = 0 \qquad t=k+R$$





$$(1-r)\bar{\mathbf{u}} \le \mathbf{u}(i) \le (1+r)\bar{\mathbf{u}} \qquad i=k,\ldots,k+R$$

$$-r^{*}\mathbf{u}(i) \le \Delta\mathbf{u}(i+1) \le r^{*}\mathbf{u}(i) \qquad i=k+1,\ldots,k+R$$

$$\Delta\mathbf{T}_{min}^{t,*} \le \left\{ \mathbf{T}^{t}(t) - \bar{\mathbf{T}}^{t} \right\} \le \Delta\mathbf{T}_{max}^{t,*} \qquad t \in [k,\ldots,k+R]$$

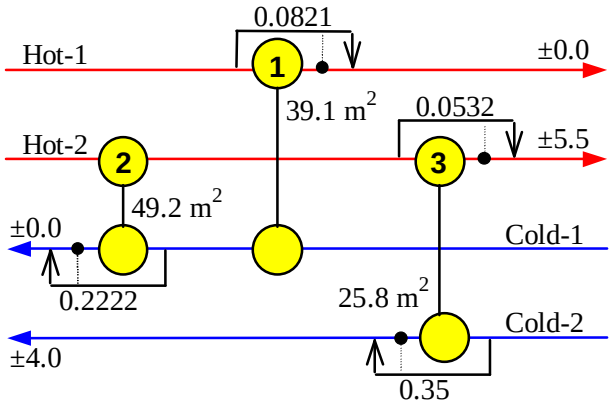
$$\dot{x}(t) = \left\| min\left\{0, \left(\mathbf{T}^{t}(t) - \bar{\mathbf{T}}^{t} + \Delta\mathbf{T}_{min}^{t,*}\right)\right\} \right\| + \left\| min\left\{0, \left(\bar{\mathbf{T}}^{t} - \mathbf{T}^{t}(t) + \Delta\mathbf{T}_{max}^{t,*}\right)\right\} \right\| \qquad t \in [k,\ldots,k+R]$$

Constraints that are aggregated

PROBLEM DIMENSIONS FOR NMPC OF RETROFIT HEN

GENERAL

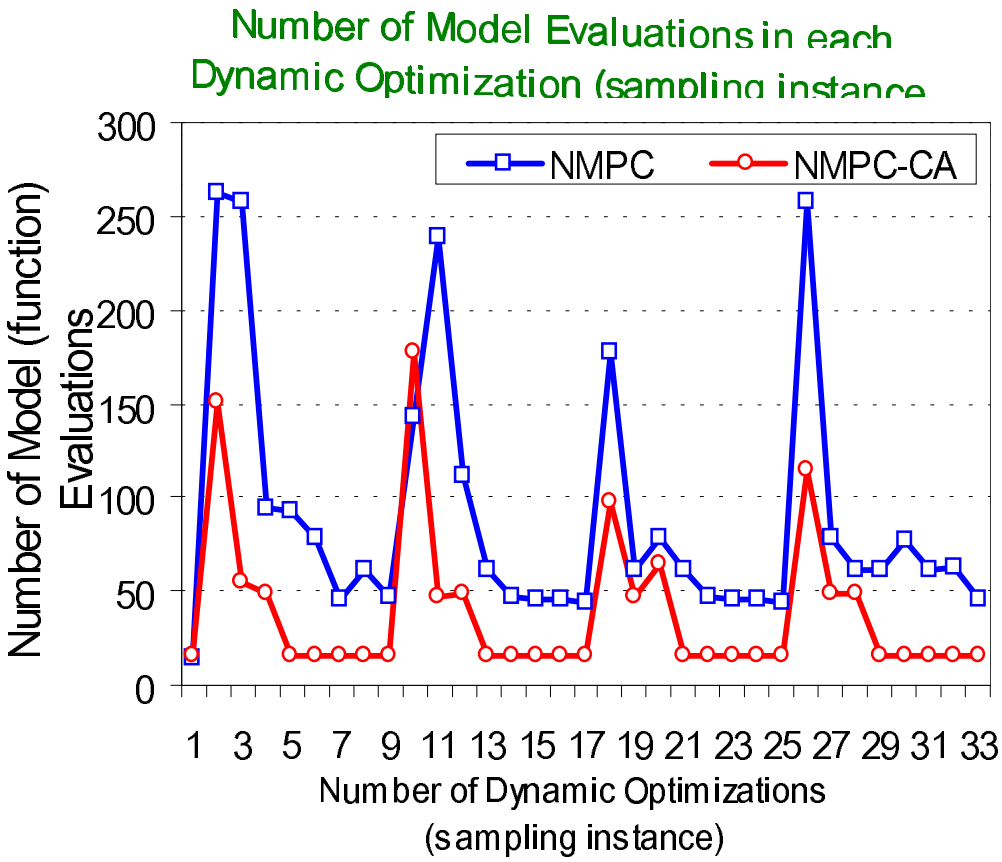
Number of target streams (**controlled variables**) = **4**
Number of inputs (**manipulated variables**) = **4**
Number of optimization **decision variables** = **12**
Number of **PDEs** = **6**
Number of **ODEs** after discretization = **146**

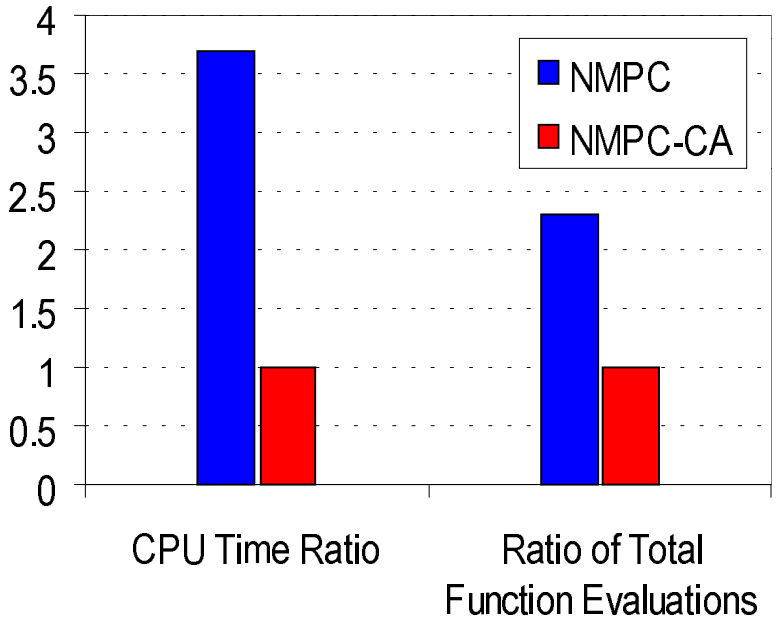


Constraints		NMPC	NMPC with CA
Number of **linear inequality** constraints	**Ax–b ≤ 0**	24	–
Number of **nonlinear inequality** constraints	**g(x) ≤ 0**	–	1
Number of **linear equality** constraints	**Ax–b = 0**	–	–
Number of **nonlinear equality** constraints	**h(x) = 0**	8	–
TOTAL		32	1

OPTIMIZER : ADS v1.10 (G.N. Vanderplaats, 1984)
METHODs : Sequential unconstrained minimization with exterior penalty function (**SUMT**)
Broydon-Fletcher-Goldfarb-Shanno (BFGS) variable metric method
One-dimensional search : polynomial interpolation after finding bounds
ISRAT = 1, IOPT = 3, IONED = 3

COMPARISON OF NMPC & NMPC-CA





CONCLUSIONS

- The use of the constraint aggregation technique in generic NMPC reduces the total number of constraints (manipulated-variable bounds, velocity constraints, state inequality and state path constraints --as terminal time equality constraints--) into one nonlinear inequality constraint.
- NMPC with constraint aggregation significantly reduces the number of total and per-step function evaluations, and thus, provides significant savings in the total computation time.
- NMPC-CA, therefore, allows the use of shorter sampling time intervals, which increases the effectiveness and the stability of the overall controller.
- However, there are numerical problems, such as the “exponential overflow”, associated with the highly ill-conditioned aggregation function (nonlinear inequality). The performance and the robustness of the technique can be further increased by elimination of numerical problems.
- Different optimization algorithms show different sensitivities to the highly nonlinear aggregation function. Compared to the SUMT method used in this work, the gain due to the use single nonlinear inequality constraint is expected to be more pronounced under SQP scheme, which evaluates the partial derivatives of all (nonlinear) constraints; while consuming significant portion of the CPU time.

Acknowledgements

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